ON FUNCTION THEORY IN QUANTUM DISC: q-DIFFERENTIAL EQUATIONS AND FOURIER TRANSFORM

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1 Green function and Poisson equation

It was shown in [7] that the Laplace-Beltrami operator $\Box : L^2(d\nu)_q \to L^2(d\nu)_q$ has a bounded inverse. Hence, for any function $f \in L^2(d\nu)_q$, there exists a unique solution $u \in L^2(d\nu)_q$ of Poisson equation $\Box u = f$.

Proposition 1.1 $\Box^{-1} f_0 = -(1-q^2)\sum\limits_{m=1}^{\infty}\dfrac{q^{-2}-1}{q^{-2m}-1}(1-zz^*)^m$.

Proof. It was shown in [7, section 5] that the 'radial part' $\Box^{(0)} : L^2(d\nu)_q \to L^2(d\nu)_q$ of the Laplace-Beltrami operator $\Box$ is given by $\Box^{(0)} = Dx(q^{-1}x-1)D$, with $x = (1-zz^*)^{-1}$. Hence, $\Box^{-1} f_0 = \psi(x)$,

$$\begin{cases} x(q^{-1}x-1)D\psi(x) = q^{-1}-q \\ \sum\limits_{j=0}^{\infty}|\psi(q^{-2j})|^2 \cdot q^{-2j} < \infty \end{cases}. \tag{1.1}$$

Thus, for all $x \in q^{-2\mathbb{Z}_+}$ one has

$$(q^{-2}x-1)(\psi(q^{-2}x)-\psi(x)) = (q^{-1}-q)^2,$$

$$\psi(x) = \psi(q^{-2}x) - (q^{-2}-1)^2\frac{q^4x^{-1}}{1-q^2x^{-1}}. \tag{1.2}$$

Now use (1.1) and (1.2) to get

$$\psi(x) = -(q^{-2}-1)^2q^2\sum_{j=1}^{\infty}\frac{q^{2j}x^{-1}}{1-q^{2j}x^{-1}} = -(q^{-2}-1)^2q^2\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}q^{2jm}x^{-m} =$$

$$= -(q^{-2}-1)^2q^2\sum_{m=1}^{\infty}\frac{q^{2m}}{1-q^{2m}}(1-zz^*)^m. \qquad \Box$$

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Consider the integral operator $I_m : D(U)_q \to D(U)'_q$ with the kernel $G_m \in D(U \times U)'_q$ given by

$$G_m = \{\Big((1-\zeta\zeta^*)(1-z^*\zeta)^{-1}\Big)^m \cdot \Big((1-z^*z)(1-z\zeta^*)^{-1}\Big)^m\}. \qquad (1.3)$$

The following statement was announced in [6, Theorem 3.5]

Theorem 1.2 *For all $f \in D(U)_q$*

$$\Box^{-1} f = -\sum_{m=1}^{\infty} \frac{q^{-2}-1}{q^{-2m}-1} I_m f. \qquad (1.4)$$

To prove this theorem we need the following auxiliary result

Lemma 1.3 *G_m is an invariant of the $U_q\mathfrak{sl}_2$-module $D(U \times U)'_q$.*

Proof of lemma. The following invariants were introduced in [8]:

$$k_{22}^{-m} k_{11}^{-m} =$$

$$= q^{2m}\left\{(1-\zeta\zeta^*)^m \cdot \sum_{j=0}^{\infty} \frac{(q^{2m};q^2)_j}{(q^2;q^2)_j}(q^{2(-m+1)}z^*\zeta)^j \cdot \sum_{n=0}^{\infty} \frac{(q^{2m};q^2)_n}{(q^2;q^2)_n}(q^{-2m}z\zeta^*)^m(1-z^*z)^m\right\}.$$

By a virtue of the q-binomial theorem (see [3]),

$$\sum_{i=0}^{\infty} \frac{(q^{2m};q^2)_i}{(q^2;q^2)_i} t^i = {}_1\Phi_0(q^{2m};-;q^2,t) = (q^{2m}t;q^2)_\infty/(t;q^2)_\infty = (t;q^2)_m^{-1}.$$

Hence,

$$k_{22}^{-m} k_{11}^{-m} = q^{2m}\left\{(1-\zeta\zeta^*)^m (q^{-2(m-1)}z^*\zeta;q^2)_m^{-1} \cdot (q^{-2m}z\zeta^*;q^2)_m^{-1} \cdot (1-zz^*)^m\right\}.$$

On the other hand, in $\mathrm{Pol}(\mathbb{C})_q$ one has $(1-\zeta\zeta^*)\zeta = q^2\zeta(1-\zeta\zeta^*)$, and in $\mathrm{Pol}(\mathbb{C})_q^{\mathrm{op}}$, respectively, $z(1-z^*z) = q^2(1-z^*z)z$, whence

$$k_{22}^{-m} k_{11}^{-m} = q^{2m}\left\{((1-\zeta\zeta^*)(1-z^*\zeta)^{-1})^m((1-z^*z)(1-z\zeta^*)^{-1})^m\right\}.$$

The invariance of G_m follows from the invariance of $k_{22}^{-m}k_{11}^{-m}$. □

Proof of theorem 1.2. In the special case $f = f_0$ one has $I_m f_0 = (1-q^2)(1-zz^*)^m$ since $\zeta^* f_0 = f_0\zeta = 0$, $\displaystyle\int_{U_q} f_0 d\nu = 1-q^2$. Hence in that special case (1.4) follows from proposition 1.1.

By [7, proposition 3.9] f_0 generates the $U_q\mathfrak{sl}_2$-module $D(U)_q$. What remains is to show that the linear operators $\Box^{-1}$ and $-\displaystyle\sum_{m=1}^{\infty}\frac{q^{-2}-1}{q^{-2m}-1}I_m$ are morphisms of $U_q\mathfrak{sl}_2$-modules. For the first operator this follows from [7, proposition 4.3] and for the second one from lemma 1.3. □

2 Cauchy-Green formula

Let $f \in D(U)_q$. This section presents a solution of the $\overline{\partial}$-problem in $L^2(d\mu)_q$:

$$\frac{\partial^{(r)}}{\partial z^*} u = f, \qquad u \perp \mathrm{Ker}\left(\frac{\partial^{(r)}}{\partial z^*}\right). \tag{2.1}$$

Our aim is to prove the following statement (see [6, proposition 4.1])

Theorem 2.1. *Let $f \in D(U)_q$. Then*

1. *There exists a unique solution $u \in L^2(d\mu)_q$ of the $\overline{\partial}$-problem $\overline{\partial} u = f$, which is orthogonal to the kernel of $\overline{\partial}$.*
2. $u = \frac{1}{2\pi i} \int\limits_{U_q} d\zeta \frac{\partial^{(l)}}{\partial z} G(z,\zeta) f d\zeta^*$, *with $G \in D(U \times U)'_q$ being the Green function of the Poisson equation.*
3. $f = -\frac{1}{2\pi i} \int\limits_{U_q} (1 - z\zeta^*)^{-1}(1 - q^{-2} z\zeta^*)^{-1} d\zeta f(\zeta) d\zeta^* - \frac{1}{2\pi i} \int\limits_{U_q} d\zeta \frac{\partial^{(l)}}{\partial z} G(z,\zeta) \cdot \frac{\partial^{(r)} f}{\partial \zeta^*} d\zeta^*.$

To clarify the symmetry of this problem, pass from the partial derivative to the differential, and from functions to differential forms.

Consider the morphism of $U_q\mathfrak{su}(1,1)$-modules $\overline{\partial} : \Omega(U)_q^{(1,0)} \to \Omega(U)_q^{(1,1)}$. By a virtue of the canonical isomorphisms of covariant $D(U)_q$-bimodules $\Omega(U)_{-2,q}^{(0,j)} \simeq \Omega(U)_q^{(1,j)}$, $j = 0, 1$, $f v_{-2} \mapsto f dz$, $f \in \Omega(U)_q^{(0,*)}$, the following scalar products are $U_q\mathfrak{su}(1,1)$-invariant (see [7]):

$$(f_1 dz, f_2 dz) = \int\limits_{U_q} f_2^* f_1 (1 - zz^*)^2 d\nu, \qquad (f_1 dz dz^*, f_2 dz dz^*) = \int\limits_{U_q} f_2^* f_1 (1 - zz^*)^4 d\nu.$$

The completions of pre-Hilbert spaces $\Omega(U)_q^{(1,0)}$, $\Omega(U)_q^{(1,1)}$, are canonically isomorphic to the Hilbert spaces $L^2(d\mu)_q$, $L^2((1 - zz^*)^2 d\mu)_q$, respectively ($i_0 : f dz \mapsto f;\ \ i_1 : f dz dz^* \mapsto f$ are just those isomorphisms).

We may reduce solving the problem (2.1) to solving the following problem:

$$\overline{\partial} u = f dz dz^*, \qquad u \perp \mathrm{Ker}(\overline{\partial}), \tag{2.2}$$

where the orthogonality means that the above invariant scalar product in the space of $(1,0)$-forms vanishes.

To solve this problem, we need auxiliary linear operators $\overline{\partial}^*$, $\square^{(1,1)} = -\overline{\partial} \cdot \overline{\partial}^*$. Turn to studying these operators.

Lemma 2.2 *For all $f \in D(U)_q$, $\frac{\partial^{(l)} f^*}{\partial z^*} = \left(\frac{\partial^{(r)} f}{\partial z}\right)^*$.*

Proof. $dz^* \cdot \frac{\partial^{(l)} f^*}{\partial z^*} = \overline{\partial} f^* = (\partial f)^* = \left(\frac{\partial^{(r)} f}{\partial z} \cdot dz\right)^* = dz^* \cdot \left(\frac{\partial^{(r)} f}{\partial z}\right)^*.$ □

Lemma 2.3 *For all $f_1, f_2 \in D(U)_q$, $\left(\overline{\partial}(f_1 dz), f_2 dz dz^*\right) = \left(f_1 dz, q^2 \frac{\partial^{(r)}}{\partial z}(f_2 \cdot (1 - zz^*)^2) dz\right)$.*

Proof. An application of lemma 2.2 and the q-analogue of Green's formula (see appendix in [6]) allows one to get for all $f_1, f_2 \in D(U)_q$:

$$\left(\overline{\partial}(f_1 dz), f_2 dz dz^*\right) = -q^2 \int\limits_{U_q} f_2^* \frac{\partial^{(r)} f_1}{\partial z^*}(1 - zz^*)^2 d\mu = -q^2 \int\limits_{U_q} (1 - zz^*)^2 f_2^* \frac{\partial^{(r)} f_1}{\partial z^*} d\mu =$$

$$= \frac{q^2}{2i\pi} \int\limits_{U_q} dz (1 - zz^*)^2 f_2^* \overline{\partial} f_1 = \frac{-q^2}{2i\pi} \int\limits_{U_q} dz \overline{\partial}((1 - zz^*)^2 f_2^*) f_1 = q^2 \int\limits_{U_q} \frac{\partial^{(l)}}{\partial z^*}((1 - zz^*)^2 f_2^*) f_1 d\mu =$$

$$= q^2 \int\limits_{U_q} \left(\frac{\partial^{(r)}}{\partial z}\left(f_2 (1 - zz^*)^2\right)\right)^* f_1 d\mu = q^2 \left(f_1 dz, \frac{\partial^{(r)}}{\partial z}(f_2 \cdot (1 - zz^*)^2) dz\right). \qquad \square$$

Corollary 2.4 *The linear operator*

$$\overline{\partial}^* : \Omega(U)_q^{(1,1)} \to \Omega(U)_q^{(1,0)}; \qquad \overline{\partial}^* : f dz dz^* \mapsto q^2 \frac{\partial^{(r)}}{\partial z}(f \cdot (1 - zz^*)^2) dz,$$

is a morphism of $U_q\mathfrak{sl}_2$-modules.

Corollary 2.5 *The linear operator $\square^{(1,1)} : \Omega(U)_q^{(1,1)} \to \Omega(U)_q^{(1,1)}$ given by $\square^{(1,1)} : f dz dz^* \mapsto q^4 \frac{\partial^{(r)}}{\partial z^*} \frac{\partial^{(r)}}{\partial z}(f (1 - zz^*)^2) dz dz^*$, $f \in D(U)_q$, is an endomorphism of $U_q\mathfrak{sl}_2$-modules.*

The relation $\square^{(1,1)} = -\overline{\partial} \cdot \overline{\partial}^*$ allows one to get a solution of the $\overline{\partial}$-problem in the form $u = -\overline{\partial}^* \omega$, with ω being a solution of the Poisson equation $\square^{(1,1)} \omega = f dz dz^*$.

Find a solution of the latter equation.

Lemma 2.6 *The elements $\{z^m\}_{m>0}$, $z^* z$, $\{z^{*m}\}_{m>0}$, generate the $U_q\mathfrak{sl}_2$-module $\mathrm{Pol}(\mathbb{C})_q$.*

Proof reduces to reproducing the argument used while proving [7, theorem 3.9]. $\square$

Lemma 2.7 *The linear operator $\square' : D(U)_q' \to D(U)_q'$ given by $\square' : f \mapsto q^4 \left(\frac{\partial^{(r)}}{\partial z^*} \frac{\partial^{(r)}}{\partial z} f\right)(1 - zz^*)^2$, is an endomorphism of the $U_q\mathfrak{sl}_2$-module $D(U)_q'$.*

Proof. Consider the isomorphism of $U_q\mathfrak{sl}_2$-modules $i : \Omega(U)_q^{(0,0)} \to \Omega(U)_q^{(1,1)}$ given by $i : f \mapsto f \cdot (1 - zz^*)^{-2} dz dz^*$. Obviously, $\square' = i^{-1} \square^{(1,1)} i$. What remains is to refer to corollary 2.5. $\square$

Proposition 2.8 $q^2 \square = \square'$.

Before proving this proposition, we deduce its corollaries.

Corollary 2.9 $\Box^{(1,1)} = q^2 i \Box i^{-1}$.

Corollary 2.10 $\Box f = q^2 \left(\frac{\partial^{(r)}}{\partial z^*} \frac{\partial^{(r)}}{\partial z} f \right) (1 - zz^*)^2$, $f \in D(U)'_q$.

Since i is an isometry, and $0 < c_1 \leq -\Box \leq c_2$ (see [7]), one has

Corollary 2.11 $0 < c_1 \leq q^{-2} \overline{\partial} \cdot \overline{\partial}^* \leq c_2$.

Note that we have proved the boundedness of the linear map $\overline{\partial}^*$ from the completion of $\Omega(U)_q^{(1,1)}$ to the completion of $\Omega(U)_q^{(1,0)}$.

Proof of proposition 2.8. Let $f = z^* z$. By a virtue of [7, lemma 5.1] one has $\Box(z^* z) = -q^2 \Box x^{-1} = q^2 (1 - zz^*)^2 = q^{-2} \Box' f$. Thus, the relation $\Box f = q^{-2} \Box' f$ is proved in the special case $f = z^* z$. In the two another special cases $f \in \{z^m\}_{m \geq 0}$, $f \in \{z^{*m}\}_{m \geq 0}$ the above relation follows from $\Omega f = \Box f = \Box' f = 0$, with Ω being the Casimir element (see [7]). Hence, by virtue of lemmas 2.6, 2.7, the relation $\Box f = q^{-2} \Box' f$ is valid for all the polynomials $f \in \mathrm{Pol}(\mathbb{C})_q$. What remains is to apply the continuity of the linear maps $\Box$, $\Box'$ in the topological vector space $D(U)'_q$ together with the density of $\mathrm{Pol}(\mathbb{C})_q$ in $D(U)'_q$. $\Box$

The following result, together with its proof attached below, are due to S. Klimek and A. Lesniewski [4].

Proposition 2.12 *Consider the orthogonal projection P from $L^2(d\mu)_q$ onto the subspace $H^2(d\mu)_q$ generated by the monomials $\{z^m\}_{m>0}$. For all $f \in D(U)_q$ one has $Pf = \int\limits_{U_q} (1 - z\zeta^*)^{-1} (1 - q^2 z \zeta^*)^{-1} f(\zeta) d\mu(\zeta)$.*

Proof. An application of [6, lemma 7.1] and the q-binomial theorem (see [3]) yield the following explicit expression for the kernel of the integral operator P:

$$\sum_{m=0}^{\infty} \frac{(q^4; q^2)_m}{(q^2; q^2)_m} (z\zeta^*)^* = (q^4 z \zeta^*; q^2)_\infty \cdot (z\zeta^*; q^2)_\infty^{-1}. \qquad \Box$$

Remark 2.13 Another proof of proposition 2.12, which involves no properties of q-special functions, will be presented in appendix of [9].

Proof of theorem 2.1. By corollary 2.10, $\Omega(U)_q^{(1,1)}$ contains a unique solution ω of the Poisson equation $\Box^{(1,1)} \omega = f dz dz^*$. It is given by

$$\omega = q^{-2} \left(\int\limits_{U_q} G(z, \zeta) f(\zeta) (1 - \zeta\zeta^*)^2 d\nu \right) (1 - zz^*)^{-2} dz dz^*,$$

with $G \in D(U \times U)'_q$, $G = -\sum_{m=1}^{\infty} \frac{q^{-2}-1}{q^{-2m}-1} G_m$, being the Green function found in section 1.

By lemma 2.3 and corollary 2.11, the $(1,0)$-form $\left(-\int_{U_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} f(\zeta) d\mu \right) dz$ is a solution of the $\overline{\partial}$-problem (2.2). Hence, the function $u = -\int_{U_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} f(\zeta) d\mu$ is a solution of the $\overline{\partial}$-problem (2.1). Since the uniqueness of a solution of this $\overline{\partial}$-problem is obvious, we have proved the first two statements of theorem 2.1.

Let $f \in D(U)_q$, and $u = -\int_{U_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} \frac{\partial^{(r)} f}{\partial \zeta^*} d\mu$ be the above solution of the $\overline{\partial}$-problem $\frac{\partial^{(r)} u}{\partial z^*} = \frac{\partial^{(r)} f}{\partial z^*}$, $u \perp \operatorname{Ker}\left(\frac{\partial^{(r)}}{\partial z^*} \right)$. Then $u \perp H^2(d\mu)_q$, $f - u \in H^2(d\mu)_q$, and hence $Pf = P(f-u) = f-u$. Thus, $f = u + Pf$, and by a virtue of proposition 2.12,

$$f = \int_{U_q} (1 - z\zeta^*)^{-1} (1 - q^2 z\zeta^*)^{-1} f(\zeta) d\mu(\zeta) - \int_{U_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} \frac{\partial^{(r)} f}{\partial \zeta^*} d\mu.$$

This relation implies the third statement of theorem 2.1, the Green formula.

3 Eigenfunctions of the operator $\Box$

It follows from [7, section 5] that $q\Box f = \Omega f$, $f \in D(U)'_q$, with $\Omega \in U_q\mathfrak{sl}_2$ being the Casimir element. Our purpose is to produce distributions $f \in D(U)'_q$ for which $\Omega f = \lambda f$ for some $\lambda \in \mathbb{C}$. More exactly, we shall prove the following result (it was announced in [6, proposition 5.1])

Theorem 3.1. *For all $f \in \mathbb{C}[\partial U]_q$ the element*

$$u = \int_{\partial U} P_{l+1}(z, e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi} \tag{5.3}$$

of $D(U)'_q$ is an eigenvector of $\Box$:

$$\Box u = \lambda(l) u, \quad \lambda(l) = -\frac{(1-q^{-2l})(1-q^{2l+2})}{(1-q^2)^2}.$$

We start with a similar problem for the quantum cone and, as in [8], consider the spaces $F(\widetilde{\Xi})_q^{(l)} \subset F(\widetilde{\Xi})_q$ of degree $2l$ homogeneous functions on the quantum cone $\widetilde{\Xi}$. Impose also the notation $F(\Xi)_q = F(\widetilde{\Xi})_q \cap D(\Xi)'_q$, $F(\Xi)_q^{(l)} = F(\widetilde{\Xi})_q^{(l)} \cap D(\Xi)'_q$, $l \in \mathbb{C}$.

By the construction, $F(\Xi)_q^0$ is a covariant $*$-algebra. We intend to give its description in terms of generators and relations.

Proposition 3.2 *The bilateral ideal $J \subset \mathrm{Pol}(\mathbb{C})_q$ generated by the single element $1 - zz^* = 0$, is a $U_q\mathfrak{sl}_2$-submodule of the $U_q\mathfrak{sl}_2$-module $\mathrm{Pol}(\mathbb{C})_q$.*

Proof is derivable from the explicit formulae

$$Hz = 2z, \qquad X^- z = q^{1/2}, \qquad X^+ z = -q^{-1/2} z^2, \tag{3.1}$$

$$Hz^* = -2z, \qquad X^+ z^* = q^{-1/2}, \qquad X^- z^* = -q^{1/2} z^{*2} \qquad \square \tag{3.2}$$

Corollary 3.3 *The $*$-algebra $\mathbb{C}[\partial U]_q \simeq \mathrm{Pol}(\mathbb{C})_q / J$ considered in [6] is a covariant $*$-algebra.*

Remind that in $\mathbb{C}[\partial U]_q$ one has $zz^* = z^* z = 1$.

An application of the relation (1.3) of [8] yields

Proposition 3.4 *The covariant $*$-algebra $\mathbb{C}[\partial U]_q$ is isomorphic to the covariant $*$-algebra $F(\Xi)_q^0$ as follows:*

$$i_0 : \mathbb{C}[\partial U]_q \to F(\Xi)_q^0, \qquad i_0 : z \mapsto q t_{11} t_{12}^{-1}, \qquad i_0 : z^* \mapsto t_{21}^{-1} t_{22}.$$

Note that the vector spaces $F(\Xi)_q^{(l)}$, $l \in \mathbb{C}$, are covariant $F(\Xi)_q^{(0)}$-bimodules, and the vector space $F(\Xi)_q$ is a covariant $*$-algebra. We identify the elements of $\mathbb{C}[\partial U]_q$ and their images under the embedding $i : \mathbb{C}[\partial U]_q \hookrightarrow F(\Xi)_q$.

Let $l \in \mathbb{C}$, $x = t_{12} t_{12}^* = -q t_{12} t_{21}$. Apply the relation (1.3) of [8] to get a description of the covariant bimodule $F(\Xi)_q^{(l)}$.

Proposition 3.5 *For all $l \in \mathbb{C}$, $x^l \in F(\Xi)_q^{(l)}$ one has*

$$zx^l = q^{2l} xz, \qquad z^* x^l = q^{-2l} x^l z^* \tag{3.3}$$

$$\begin{cases} X^+(x^l) = q^{-3/2} \dfrac{q^{-2l} - 1}{q^{-2} - 1} z x^l \\ X^-(x^l) = q^{3/2} \dfrac{1 - q^{2l}}{1 - q^2} z^* x^l \\ H(x^l) = 0 \end{cases} \tag{3.4}$$

The covariant bimodules $F(\Xi)_q^{(l)}$ are, in particular, $U_q\mathfrak{sl}_2$-modules. The associated representations of $U_q\mathfrak{sl}_2$ are called the representations of the principal series. These are irreducible for some open dense set of $l \in \mathbb{C}$. By a virtue of relation (5.8) of [7], for those $l \in \mathbb{C}$, and hence for all $l \in \mathbb{C}$ and all $f \in F(\Xi)_q^{(l)}$, one has

$$\Omega f = \Lambda(l) f, \qquad \Lambda(l) = \frac{(q^{-l} - q^l)(q^{-(l+1)} - q^{l+1})}{(q^{-1} - q)^2}. \tag{3.5}$$

Let $V^{(l)}$ be the $U_q\mathfrak{sl}_2$-modules considered in [7]. One can easily deduce from (3.1), (3.4), (3.5) the following

Corollary 3.6 *For all $l \in \mathbb{C}$, the linear map $i_l : V^{(l)} \to F(\Xi)_q^{(l)}$; $i_l : X^{\pm m} e_0 \mapsto X^{\pm m}(x^l)$, $m \in \mathbb{Z}_+$, are the isomorphisms of $U_q\mathfrak{sl}_2$-modules.*

Proof of theorem 3.1. Let us turn to a construction of distributions $f \in D(X)'_q$ on the quantum hyperboloid, which satisfy the equation $\Omega f = \Lambda(l) f$ for some $l \in \mathbb{C}$.

By the results of [8, section 6], the element

$$k_{22}^l k_{11}^l \overset{\mathrm{def}}{=} q^{-2l}\xi^l \sum_{j=0}^{\infty} \frac{(q^{-2l};q^2)_j}{(q^2;q^2)_j}(q^{2(l+1)}z^*\zeta)^j \cdot \sum_{m=0}^{\infty} \frac{(q^{-2l};q^2)_m}{(q^2;q^2)_m}(q^{2l}z\zeta^*)^m(1-z^*z)^{-l}$$

of the completion of $F(X)^{\mathrm{op}} \otimes F(\Xi)_q^{(l)}$ is an invariant. (Here $z, z^* \in F(X)^{\mathrm{op}}$, $\zeta, \zeta^*, \xi \in F(\Xi)_q$ are the elements given by explicit formulae in [8, section 6]).

It follows from the results of [8, section 4] that the linear functional $\eta : F(\Xi)_q^{(-1)} \to \mathbb{C}$,

$$\int_{\Xi_q} \left(\sum_{m=-\infty}^{\infty} a_m \zeta^m \right) \xi^{-1} d\eta = a_0,$$

is an invariant integral. Hence, the linear integral operator

$$F(\Xi)_q^{(-l-1)} \to F(X)_q; \qquad f \mapsto \int_{\Xi_q} \{k_{22}^l k_{11}^l\} f d\eta$$

is a morphism of $U_q\mathfrak{sl}_2$-modules. By a virtue of (3.5), for any trigonometric polynomial $f(\zeta) \in \mathbb{C}[\partial U]_q$, the function

$$\int_{\Xi_q} \{k_{22}^l k_{11}^l\} f \xi^{-(l+1)} d\eta = q^{-2l} \int_{\partial U} P_{-l}(z, e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}$$

is an eigenfunction of the Laplace-Beltrami operator. Here P_{-l} is a q-analogue of the Poisson kernel (see [6, section 5]). Now a passage from the quantum hyperboloid X to the quantum disc U via the isomorphism of $U_q\mathfrak{sl}_2$-modules $i : D(U)'_q \overset{\rightarrow}{\approx} D(X)'_q$ (see [8]) yields the statement of theorem 3.1 □

Denote by $\mathbb{C}[\partial U]_{q,l}$ the vector space $\mathbb{C}[\partial U]_q$ equipped by the structure of $U_q\mathfrak{sl}_2$-module in such a way that the map $\mathbb{C}[\partial U]_{q,l} \to F(\Xi)_q^{(l)}$; $f(z) \mapsto f(z)x^l$, is a morphism of $U_q\mathfrak{sl}_2$-modules.

An application of (3.1), (3.4) gives

$$X^+ f(z) = -q^{-1/2} z^2 (Df)(z) + q^{-3/2} \frac{q^{-2l}-1}{q^{-2}-1} f(qz),$$

$$X^- f(z) = q^{1/2}(Df)(z) + q^{3/2} \frac{1-q^{2l}}{1-q^2} f(qz),$$

$$Hf(z) = 2z \frac{d}{dz} f(z),$$

with $D : f(z) \mapsto (f(q^{-1}z) - f(qz))/(q^{-1}z - qz)$, $f \in \mathbb{C}[\partial U]_{q,l}$.

Let $\operatorname{Re} l > -\frac{1}{2}$. With the notation of [6] being implicit, introduce a linear operator I_l in $\mathbb{C}[\partial U]_{q,l}$ given by

$$I_l f = \frac{\Gamma_{q^2}^2(l+1)}{\Gamma_{q^2}(2l+1)} \lim_{\substack{1-r^2 \in q^{2\mathbb{Z}_+} \\ r \to 1}} (1-r^2)^l b_r u, \tag{3.6}$$

with $u=\int\limits_{\partial U}P_{l+1}(z,e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi}$, $f\in\mathbb{C}[\partial U]_{q,l}$.

Our aim now is to prove the following result (see [6, proposition 5.3])

Theorem 3.7. $I_l f=f$

Proof. The theorem will be proved if we establish the existence of the limit in the right hand side of (3.6) and show that I_l is the identity operator.

Let $L\subset\mathbb{C}[\partial U]_{q,l-1}$ be the linear subspace of all those elements $f\in\mathbb{C}[\partial U]_q$ for which the both above statements are valid. By a virtue of [6, lemma 5.4],

$$\lim_{\substack{x\in q^{-2\mathbb{Z}_+}\\ z\to+\infty}}\varphi_l\left(\frac{1}{x}\right)\Bigg/\left(\frac{\Gamma_{q^2}(2l+1)}{\Gamma_{q^2}^2(l+1)}x^l\right)=1, \tag{3.7}$$

for $\mathrm{Re}\,l>-\frac{1}{2}$, with $\varphi_l=\int\limits_{\partial U}P_{l+1}(z.e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi}$. Thus, $1\in L$. Moreover, an application of this lemma and the fact that the linear operator

$$j_l:\mathbb{C}[\partial U]_{q,l}\to D(U)_q';\qquad j_l:f\mapsto\int\limits_{\partial U}P_{l+1}(z,e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi}$$

is an isomorphism of $U_q\mathfrak{sl}_2$-modules, allows one to prove that L is a submodule of the $U_q\mathfrak{sl}_2$-module $\mathbb{C}[\partial U]_{q,l}$. On the other hand, with $l\notin\mathbb{Z}_++\frac{\pi}{\ln(q^{-1})}\mathbb{Z}$, the $U_q\mathfrak{sl}_2$-module $\mathbb{C}[\partial U]_{q,l}\simeq V^{(l)}$ is simple. Hence, for l as above one has $L=\mathbb{C}[\partial U]_{q,l}$, and thus the theorem is proved.□

REMARK 3.8. Let $m\in\mathbb{Z}_+$, and $\psi(x)$ be a function on $q^{-2\mathbb{Z}_+}$ such that $z^m\cdot\psi(y^{-1})=\int\limits_{\partial U}P_{l+1}(z,e^{i\theta})e^{im\theta}\frac{d\theta}{2\pi}$. Another way of proving the existence of the limit in the right hand side of (3.6) is based on producing a fundamental system of solutions of the difference equation for $\psi(x)$. (This difference equation is a consequence of the relation $\Omega(z^m\psi(y^{-1}))=\Lambda(l)(z^m\psi(y^{-1}))$.) It is easy to prove the existence of such fundamental system of solutions ψ_1,ψ_2 that

$$\lim_{x\to+\infty}\frac{\psi_1(x)}{x^l}=\lim_{x\to+\infty}\frac{\psi_2(x)}{x^{-l-1}}=1.$$

What remains is to use the relation $\mathrm{Re}\,l>-\frac{1}{2}$.

4 Decomposing in eigenfunctions of the operator $\square^{(0)}$

One can find in [7, section 5] a description of the bounded linear operator $\square^{(0)}:f(x)\mapsto Dx(q^{-1}x-1)Df(x)$ in the Hilbert space $L^2(d\nu)_q^{(0)}$ of such functions on $q^{-2\mathbb{Z}_+}$ that $\|f\|=\left(\int\limits_1^\infty|f(x)|^2d_{q^{-2}}x\right)^{1/2}<\infty$. That section also contains the relation (5.9) which determines the eigenfunctions $\Phi_l(x)$ of $\square^{(0)}$. Besides, a unitary operator $u:L^2(d\nu)_q^{(0)}\to L^2(dm)$ that

realizes a decomposition in those eigenfunctions was constructed. Remind that u could be defined by (5.10), and dm is a Borel measure on a compact $\mathfrak{L}_0$ introduced by (5.7) in [7].

In this section, explicit formulae for eigenfunctions $\Phi_l(x)$ and the spectral measure will be found; [6, proposition 3.2] will be proved.

Proposition 4.1 $\Phi_l(x) = {}_3\Phi_2\left[\begin{matrix} x, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix}\right]$.

Proof. By a virtue of [6, corollary 5.2], the distribution

$$ {}_3\Phi_2\left[\begin{matrix} (1-zz^*)^{-1}, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix}\right] \in D(U)'_q $$

is an eigenfunction of $\Box$. What remains is to apply the definition of $\Phi_l(x)$ and the evident relation ${}_3\Phi_2\left[\begin{matrix} 1, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix}\right] = 1$. $\Box$

Corollary 4.2 *The spectrum of* $\Box^{(0)}$ *coincides with the segment* $\left[-\frac{1}{(1-q)^2}, -\frac{1}{(1+q)^2}\right]$.

Proof. It follows from [6, section 5] that the continuous spectrum of $\Box^{(0)}$ fills this segment. So we are to prove that the discrete spectrum of $\Box^{(0)}$ is void, that is $\Phi_l \notin L^2(d\nu)_q^{(0)}$ for $\mathrm{Re}\, l > -\frac{1}{2}$. This can be deduced from proposition 5.1 and lemma 5.4 of [6]. $\Box$

By corollary 4.2, the carrier of dm coincides with the segment $\{l \in \mathbb{C} |\ \mathrm{Re}\, l = -\frac{1}{2},\ 0 \leq \mathrm{Im}\, l \leq \frac{\pi}{h}\}$, with $h = -2\ln q$. Hence,

$$ \frac{1}{(1+q)^2} \leq -\Box^{(0)} \leq \frac{1}{(1-q)^2}. $$

This inequality implies [6, proposition 3.2].

We intend to obtain an explicit formula for the kernel $G(x, \xi, l)$ of the integral operator $(\Box^{(0)} - \lambda(l)I)^{-1}$ in $L^2(d\nu)_q^{(0)}$. By corollary 4.2, the 'Green function' $G(x, \xi, l)$ is well defined and holomorphic in l for $x, \xi \in q^{-2\mathbb{Z}_+}$, $\mathrm{Re}\, l \neq -\frac{1}{2}$.

Remind the notation $[a]_q = (q^{-a} - q^a)/(q^{-1} - q)$, and choose the branch of x^l in the half-plane $\mathrm{Re}\, x > 0$: $x^l = e^{\ln x \cdot l}$, with $\ln x$ being the principal branch of the logarithm.

Lemma 4.3 *With* $|x| > q^2$, $\mathrm{Re}\, x > 0$, *the function*

$$ \psi_l(x) = x^l \cdot {}_2\Phi_1\left(\begin{matrix} q^{-2l}, q^{-2l}; q^2; q^2 x^{-1} \\ q^{-4l} \end{matrix}\right) \tag{4.1} $$

satisfies the difference equation

$$ Dx(q^{-1}x - 1)D\psi_l(x) = \lambda(l)\psi_l(x). \tag{4.2} $$

Proof. The right hand side of (4.1) is of the form $x^l \sum_{m=0}^{\infty} \frac{a_m}{x^m}$, $a_m \in \mathbb{C}$. Its substitution into (4.2) gives

$$\frac{a_{m+1}}{a_m} = q \frac{[l-m]_q^2}{[l-m]_q[l-1-m]_q - [l]_q[l+1]_q} = q \frac{[l-m]_q}{[m+1]_q[m-2l]_q} =$$

$$= q^2 \frac{(1-q^{-2l+2m})^2}{(1-q^{2(m+1)})(1-q^{-4l+2m})}.$$

What remains is to use the definition of the basic hypergeometric series ${}_2\Phi_1$ (see [3]).

This lemma and the definition of the Green function $G(z,\xi,l)$ imply

Proposition 4.4
1) For $\mathrm{Re}\, l > -\frac{1}{2}$

$$G(x,\xi,l) = c_1(l) \begin{cases} \psi_l(\xi) f_l(x), & x \le \xi \\ f_l(\xi)\psi_l(x), & x \ge \xi \end{cases} \tag{4.3}$$

2) For $\mathrm{Re}\, l < -\frac{1}{2}$

$$G(x,\xi,l) = c_2(l) \begin{cases} \psi_{-1-l}(\xi) f_l(x), & x \le \xi \\ f_l(\xi)\psi_{-1-l}(x), & x \ge \xi \end{cases} \tag{4.4}$$

Here $x, \xi \in q^{-2\mathbb{Z}_+}$, $c_1(l), c_2(l) \in \mathbb{C}$.

Find the 'constants' $c_1(l), c_2(l)$.

Lemma 4.5 *For any two functions* u, v *on the semi-axis* $x > 0$,

$$Du(x) \cdot v(x) = D(u(x)v(qx)) - qu(qx)(Dv)(qx).$$

Proof. The following q-analogue of Leibnitz formula is directly from the definition of D:

$$D(u(x)v(x)) = (Du)(x) \cdot v(q^{-1}x) + u(qx)(Dv)(x).$$

Replace $v(x)$ by $v(qx)$ to get

$$(Du)(x)v(x) = D(u(x)v(qx)) - u(qx)D(v(qx)).$$

What remains is to apply the straightforward relation $D(v(qx)) = q(Dv)(qx)$. □

Let $l \in \mathbb{C}$, $x \in q^{-2\mathbb{Z}_+}$, and $\varphi_1(x), \varphi_2(x)$ be solutions of the difference equation $Dx(q^{-1}x-1)D\varphi = \lambda(l)\varphi$.

Lemma 4.6

$$W(\varphi_1,\varphi_2) = x(q^{-2}x-1)\left(\frac{\varphi_1(q^{-2}x)-\varphi_1(x)}{q^{-2}x-x} \cdot \varphi_2(x) - \varphi_1(x)\frac{\varphi_2(q^{-2}x)-\varphi_2(x)}{q^{-2}x-x}\right)$$

does not depend on $x \in q^{-2\mathbb{Z}_+}$.

Proof. Evidently,

$$0=(Dx(q^{-1}x-1)D\varphi_1)\varphi_2-\varphi_1(Dx(q^{-1}x-1)D\varphi_2).$$

Hence, by a virtue of lemma 4.5,

$$0=D(x(q^{-1}x-1)D\varphi_1(x)\cdot\varphi_2(qx))-D(x(q^{-1}x-1)D\varphi_2(x)\cdot\varphi_1(qx)).$$

That is,

$$D(x(q^{-1}x-1)(D\varphi_1(x)\cdot\varphi_2(qx)-\varphi_1(qx)D\varphi_2(x))=0.$$

Hence, $q^{-1}x(q^{-2}x-1)((D\varphi_1(q^{-1}x)\cdot\varphi_2(x)-\varphi_1(x)(D\varphi_2)(q^{-1}x))$ is a constant. $\square$.

Let $\varphi_1(x),\varphi_2(x)$ be the eigenfunctions involved in the formulation of the previous lemma, and set up

$$\Phi(x,\xi)=\begin{cases}\varphi_1(x)\varphi_2(\xi), & x\geq\xi\\ \varphi_1(\xi)\varphi_2(x), & x\leq\xi\end{cases}.$$

Lemma 4.7

$$Dx(q^{-1}x-1)D\Phi(x,\xi)\Big|_{x=\xi}=\begin{cases}\frac{W(\varphi_1,\varphi_2)}{(1-q^2)\xi}+\lambda\Phi(\xi,\xi), & x=\xi\\ \lambda\Phi(x,\xi), & x\neq\xi\end{cases}.$$

Proof. Let $x=\xi$:

$$Dx(q^{-1}x-1)D\Phi\Big|_{x=\xi}=\left.\frac{q^{-1}x(q^{-2}x-1)\frac{\Phi(q^{-2}x,\xi)-\Phi(x,\xi)}{q^{-2}x-x}-x(x-1)\frac{\Phi(x,\xi)-\Phi(q^2x,\xi)}{x-q^2x}}{q^{-1}x-qx}\right|_{x=\xi}=$$

$$=\frac{1}{(q^{-1}-q)\xi}\left(q^{-1}\xi(q^{-2}\xi-1)\frac{\varphi_1(q^{-2}\xi)-\varphi_1(\xi)}{q^{-2}\xi-\xi}\cdot\varphi_2(\xi)-q\xi(\xi-1)\varphi_1(\xi)\frac{\varphi_2(\xi)-\varphi_2(q^2\xi)}{\xi-q^2\xi}+\right.$$
$$\left.+q^{-1}\xi(q^{-2}\xi-1)\varphi_1(\xi)\frac{\varphi_2(q^{-2}\xi)-\varphi_2(\xi)}{q^{-2}\xi-\xi}-q^{-1}\xi(q^{-2}\xi-1)\varphi_1(\xi)\frac{\varphi_2(q^{-2}\xi)-\varphi_2(\xi)}{q^{-2}\xi-\xi}\right).$$

We did not break the equality since we have added and then subtracted from its right hand side the same expression:

$$\frac{1}{(q^{-1}-q)\xi}q^{-1}\xi(q^{-2}\xi-1)\varphi_1(\xi)\frac{\varphi_2(q^{-2}\xi)-\varphi_2(\xi)}{q^{-2}\xi-\xi}.$$

Thus we get

$$Dx(q^{-1}x-1)D\Phi\Big|_{x=\xi}=\frac{q^{-1}}{(q^{-1}-q)\xi}\cdot W(\varphi_1,\varphi_2)+\lambda\cdot\Phi(\xi,\xi).$$

In the case $x\neq\xi$ the statement of the lemma is evident. $\square$.

Corollary 4.8
1) For $\mathrm{Re}\,l>-\frac{1}{2}$*,* $W(\psi_l,f_l)\neq 0$*,* $c_1(l)=\dfrac{1}{W(\psi_l,f_l)}$*,*
2) For $\mathrm{Re}\,l<-\frac{1}{2}$*,* $W(\psi_{-1-l},f_l)\neq 0$*,* $c_2(l)=\dfrac{1}{W(\psi_{-1-l},f_l)}$

Find $W(\psi_l, f_l)$, $W(\psi_{-1-l}, f_l)$ as functions of an indeterminate l. Remind the notation (see [6, section 6]):

$$c(l) = \frac{\Gamma_{q^2}(2l+1)}{\left(\Gamma_{q^2}(l+1)\right)^2} = \frac{(q^{2(l+1)}; q^2)^2_\infty}{(q^{2(2l+1)}; q^2)_\infty (q^2; q^2)_\infty}. \tag{4.5}$$

Lemma 4.9 *For all $l \notin \frac{1}{2} + \mathbb{Z}$, $f_l(x) = c(l)\psi_l(x) + c(-1-l)\psi_{-1-l}(x)$.*

Proof. Consider the functions f_l, ψ_l, ψ_{-1-l} holomorphic in the domain $l \notin \frac{1}{2} + \mathbb{Z}$. Evidently, $\{\psi_l, \psi_{-1-l}\}$ form the base in the vector space of solutions for the equation $Dx(q^{-1}x - 1)D\psi = \lambda(l)\psi$ in the space of functions on $q^{-2\mathbb{Z}_+}$. Hence $f_l(x) = a(l)\psi_l(x) + b(l)\psi_{-1-l}(x)$, with $a(l), b(l)$ being holomorphic in the domain $l \notin \frac{1}{2} + \mathbb{Z}$. Let $x \in q^{-2\mathbb{Z}_+}$ go to infinity. By a virtue of [6, lemma 5.4], $a(l) = c(l)$ for $\mathrm{Re}\, l > -\frac{1}{2}$, and $b(l) = c(-1-l)$ for $\mathrm{Re}\, l < -\frac{1}{2}$. What remains is to apply the holomorphy of $a(l), b(l), c(l), c(-1-l)$ in the domain $l \notin \frac{1}{2} + \mathbb{Z}$. □

Lemma 4.10 $W(\psi_l, \psi_{-1-l}) = [2l+1]_q$.

Proof. With $x \in q^{-2\mathbb{Z}_+}$, $x \to +\infty$, one has

$$\psi_l(x) \sim x^l, \qquad \frac{\psi_l(q^{-2}x) - \psi_l(x)}{q^{-2}x - x} \sim \frac{q^{-2l} - 1}{q^{-2} - 1} x^{l-1}.$$

Hence by lemma 4.6,

$$W(\psi_l, \psi_{-1-l}) = \lim_{\substack{x \to +\infty \\ x \in q^{-2\mathbb{Z}_+}}} x(1 - q^{-2}x) \left(\frac{q^{-2l} - 1}{q^{-2} - 1} - \frac{q^{-2(-1-l)} - 1}{q^{-2} - 1} \right) x^{-2}. \qquad \square$$

Lemmas 4.9, 4.10 and corollary 4.8 imply

Proposition 4.11 *The constants in (4.3) and (4.4) are given by*

$$c_1(l) = \frac{1}{c(-1-l)[2l+1]_q}, \qquad c_2(l) = -\frac{1}{c(l)[2l+1]_q}, \tag{4.6}$$

with $c(l)$ being the q-analogue of Harish-Chandra's c-function determined by (4.5).

The conclusion is as follows. For $\mathrm{Re}\, l \neq -\frac{1}{2}$ the operator $\Box^{(0)} - \lambda(l)I$ in the Hilbert space $L^2(d\nu)^{(0)}_q$ has a bounded inverse operator given by

$$((\Box^{(0)} - \lambda(l)I)^{-1}\psi)(x) = \int_1^\infty G(x, \xi, l)\psi(\xi) d_{q^2}\xi, \qquad \psi \in L^2(d\nu)^{(0)}_q.$$

The Green function is given by the explicit formulae (4.3), (4.4), (4.6).

Find the spectral projections of $\Box^{(0)}$.

The following well known result follows from the Stieltjes inversion formula (see [5]).

Proposition 4.12 *Let A be a bounded selfadjoint operator with simple purely continuous spectrum. For any interval (a_1, a_2) on the real axis, one has*

$$E((a,b)) = \lim_{\varepsilon\to+0} \frac{1}{2\pi i}\int\limits_{a_1}^{a_2}(R_{\lambda-i\varepsilon} - R_{\lambda+i\varepsilon})d\lambda,$$

with $R_\lambda = (A-\lambda I)^{-1}$.

Remark 4.13. There is an extension of proposition 4.12 to the case of an arbitrary selfadjoint operator (see [2, chapter 10, section 6]).

Proposition 4.14 *Let $x, \xi \in q^{-2\mathbb{Z}_+}$, $\operatorname{Re} l = -\frac{1}{2}$. Then*

$$\lim_{\varepsilon\to+0}(G(x,\xi,l+\varepsilon) - G(x,\xi,l-\varepsilon)) = \frac{f_l(\xi)f_l(x)}{c(l)c(-1-l)[2l+1]_q}. \tag{4.7}$$

Proof. In the case $x \le \xi$ one has due to (4.3), (4.4), (4.6):

$$\lim_{\varepsilon\to+0}(G(x,\xi,l+\varepsilon) - G(x,\xi,l-\varepsilon)) = \frac{1}{[2l+1]_q}\left(\frac{1}{c(l)}\psi_{-1-l}(\xi) + \frac{1}{c(-1-l)}\psi(\xi)\right)f_l(x). \tag{4.8}$$

Now (4.7) follows from (4.8) and lemma 4.9. The case $x \ge \xi$ is completely similar to the case $x \le \xi$. □

Remark 4.15. There is a natural generalization of proposition 4.14. Let $\operatorname{Re} l = -\frac{1}{2}$ and let $\gamma(\varepsilon)$ be such a parametric smooth curve on the complex plane that $\gamma(0) = l$, $\frac{d\gamma(0)}{d\varepsilon} > 0$. Then

$$\lim_{\varepsilon\to+0}(G(x,\xi,\gamma(-\varepsilon)) - G(x,\xi,\gamma(\varepsilon))) = \frac{f_l(\xi)f_l(x)}{c(l)c(-1-l)[2l+1]_q}. \tag{4.9}$$

Remind that the spectrum of $\square^{(0)}$ is simple, purely continuous and fills a segment. This segment was parametrized as follows:

$$\lambda(l) = -\frac{(1-q^{-2l})(1-q^{2l+2})}{(1-q^2)^2}, \qquad l = -\frac{1}{2} + i\rho,\ 0 \le \rho \le \frac{\pi}{h}.$$

Here, as before, $h = -2\ln q$. Note that

$$\frac{d\lambda}{dl} = \frac{1}{(1-q^2)^2}d(q^{-2l} + q^{2l+2}) = \frac{h}{(1-q^2)^2}(q^{-2l} - q^{2l+2}). \tag{4.10}$$

Apply proposition 4.12 to $\square^{(0)}$. An application of (4.9), (4.10) yields the main result of this section, which was kindly communicated to the authors by L. I. Korogodsky.

Associate to each finitely supported function $f(x)$ on $q^{-2\mathbb{Z}_+}$ the function

$$\widehat{f}(\rho) = \int\limits_1^\infty {}_3\Phi_2\left[\begin{matrix} x, q^{-2l}, q^{2(l+1)}; q^2; q^2 \\ q^2, 0 \end{matrix}\right] f(x)d_{q^2}x$$

on the segment $\left[0, \frac{\pi}{h}\right]$. Here $l = -\frac{1}{2} + i\rho$, $h = -2\ln q$.

EXAMPLE 4.16. Let $f_0(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}$. Then

$$\widehat{f}_0(\rho) = 1 - q^2. \tag{4.11}$$

Remind a well known result of operator theory ([1]):

Proposition 4.17 *Let A be a bounded selfadjoint operator with simple spectrum in a Hilbert space H, E_t the spectral measure of A, and g such a vector that the linear span of $\{A^m g\}_{m\in\mathbb{Z}}$ is dense in H. With $\sigma(t) = (E_t g, g)$, the map*

$$f(t) \mapsto \int\limits_{-\infty}^{\infty} f(t) dE_t g$$

is a unitary operator from $L^2_\sigma(-\infty, \infty)$ onto H. This unitary map sets up the equivalence of A and the multiplication operator $f(t) \mapsto tf(t)$ in $L^2_\sigma(-\infty, \infty)$.

Now one can prove the following

Proposition 4.18 *Consider a Borel measure*

$$d\sigma(\rho) = \frac{1}{2\pi} \cdot \frac{h}{1-q^2} \cdot \frac{d\rho}{c(-\frac{1}{2} + i\rho)c(-\frac{1}{2} - i\rho)} \tag{4.12}$$

on the segment $[0, \frac{\pi}{h}]$. The linear operator $f \mapsto \widehat{f}$ is extendable by a continuity up to a unitary operator $u : L^2(d\nu)_q^{(0)} \to L^2(d\sigma)$. For all $f \in L^2(d\nu)_q^{(0)}$,

$$u \cdot \Box^{(0)} f = \lambda(l) u f.$$

To conclude, note that the measure $dm(l)$ could be derived from the measure $d\sigma(\rho)$ via the substitution $l = -\frac{1}{2} + i\rho$.

5 Fourier transform

In [7, section 5] a unitary operator

$$\bar{i} : L^2(d\nu)_q \to \bigoplus \int\limits_{\mathfrak{L}_0} \overline{V}^{(l)} dm(l) \tag{5.1}$$

was constructed, with $\overline{V}^{(l)}$ being a completion of the $U_q\mathfrak{su}(1,1)$-module $V^{(l)}$, equipped with an invariant scalar product. By the results of the previous section,

$$\bigoplus \int\limits_{\mathfrak{L}_0} \overline{V}^{(l)} dm(l) \simeq \bigoplus \int\limits_{0}^{\pi/h} \overline{V}^{(-\frac{1}{2}+i\rho)} d\sigma(\rho),$$

with $d\sigma$ being the measure (4.12), and the modules $V^{(-\frac{1}{2}+i\rho)}$ could be replaced by the isomorphic modules $\mathbb{C}[z]_{q,-\frac{1}{2}+i\rho}$. The linear operator $\overline{i}$ is replaced by a completion in $L^2(d\nu)_q$ of a morphism of $U_q\mathfrak{sl}_2$-modules given by $if_0 = 1-q^2$. (This relation follows from (4.11); it determines unambiguously a morphism of $U_q\mathfrak{sl}_2$-modules by [7, proposition 3.9]).

Remind the notation (see [6]):

$$P_l^t = (q^2 z^*\zeta; q^2)_{-l} \cdot (z\zeta^*; q^2)_{-l}(1-\zeta\zeta^*)^l \in D(\Xi \times X)'_q.$$

Proposition 5.1 *For all $f \in D(U)_q$,*

$$if = \int\limits_{U_q} P^t_{\frac{1}{2}+i\rho}(z,\zeta)f(\zeta)d\nu$$

Proof. It is easy to show that for all $\rho \in [0, \frac{\pi}{h}]$ the linear integral operator $i_\rho : f \mapsto \int\limits_{U_q} P^t_{\frac{1}{2}+i\rho}(z,\zeta)f(\zeta)d\nu$ maps the vector space $D(U)_q$ into $\mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho}$. Now our statement follows from the following two lemmas.

Lemma 5.2 $i_\rho f_0 = 1-q^2$.

Proof. Apply the decomposition

$$P_l^t = \sum_{j>0} \zeta^j \cdot \psi_j(\xi) + \psi_0(\xi) + \sum_{j>0} \psi_{-j}(\xi)\zeta^j,$$

described in [6]. It is easy to show that only the term $\psi_0(\xi)$ contributes to the integral $i_\rho f_0$. On the other hand, $\psi_0(1) = 1$, $\int\limits_{U_q} 1 \cdot f_0 d\nu = 1-q^2$. □

Lemma 5.3 *The linear operator $i_\rho : D(U)_q \to \mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho}$ is a morphism of $U_q\mathfrak{sl}_2$-modules.*

Proof. Consider the integral operator

$$j_\rho : \mathbb{C}[\partial U]_{q,l} \to D(U)'_q, \qquad j_\rho : f \mapsto \int\limits_0^{\pi/h} P_{\frac{1}{2}-i\rho}(z, e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi}.$$

It is a morphism of $U_q\mathfrak{sl}_2$-modules, as it was noted in section 3. Equip the $U_q\mathfrak{su}(1,1)$-modules $D(U)_q$, $\mathbb{C}[\partial U]_{q,l}$ with invariant scalar products

$$D(U)_q \times D(U)_q \to \mathbb{C}, \qquad f_1 \times f_2 \mapsto \int\limits_{U_q} f_2^* f_1 d\nu,$$

$$\mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho} \times \mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho} \to \mathbb{C}, \qquad f_1 \times f_2 \mapsto \int\limits_{\partial U} f_2^* f_1 \frac{d\theta}{2\pi}.$$

It follows from the definitions that the integral operator with a kernel $K=\sum\limits_i k_i''\otimes k_i'$ is conjugate to the integral operator with the kernel $K^t=\sum\limits_i k_i'^*\otimes k_i''^*$. Hence $i_\rho=j_\rho^*$, and i_ρ is a morphism of $U_q\mathfrak{sl}_2$-modules since this is the property of j_ρ (see [7, section 5]). □

It follows from the proof of lemma 5.3 that $\overline{j}=\overline{i}^*$ is the integral operator

$$\overline{j}:f(e^{i\theta},\zeta)\mapsto\int\limits_0^{\pi/h}\int\limits_0^{2\pi}P_{\frac12-i\rho}(z,e^{i\theta})f(e^{i\theta},\rho)\frac{d\theta}{2\pi}d\sigma(\rho).$$

Since $\overline{i}$ is unitary (see [7]), $\overline{j}\cdot\overline{i}=\overline{i}\cdot\overline{j}=1$. Hence $\overline{i},\overline{j}$ coincide with the operators F,F^{-1} introduced in [6], respectively. This implies the statement of [6, proposition 6.1].

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